Relating vector ray-tracing equations for holograms of arbitrary shape and thickness

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The author derives Latta's ray-tracing equations for holograms of arbitrary thickness [Appl. Opt. **10**, 2698 (1971)] from Welford's vector ray-tracing equation for holograms of arbitrary shape [Opt. Commun. **14**, 322 (1975)]. The derivation follows Welford's original approach but accounts for changes in the shape and thickness of the recording medium between construction and reconstruction. © 2008 Optical Society of America

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1. INTRODUCTION

Understanding the effects that holographic recording media have on light propagation is important in the design and use of holograms, especially holographic optical elements. Ray tracing is an approach adopted from geometrical refractive optics to characterize diffraction effects in holograms. Welford [1] derived a vector ray-tracing equation applicable to holograms of arbitrary shape but restricted to infinitely thin holograms that do not change shape between the time of construction and reconstruction:

$$\hat{n} \times (\hat{o}' - \hat{r}') = m\frac{\lambda'}{\lambda}\hat{n} \times (\hat{o} - \hat{r}). \tag{1}$$

The caret denotes a unit vector, the prime denotes conditions during reconstruction, $\hat{n}$ is the normal unit vector to the point on the hologram surface where object and reference as well as output and reconstruction rays $\hat{o},\hat{r}$ and $\hat{o}',\hat{r}'$, respectively, intersect, and m is the order of diffraction (+1 for the image, −1 for the conjugate image). The magnitudes of the cross products in Eq. (1) are equal to the sines of the angles θ for each ray with respect to the local surface normal. The result is the commonly used grating equation, which characterizes the surface grating structure responsible for angular playout:

$$\sin(\theta_o') - \sin(\theta_r') = m\frac{\lambda'}{\lambda}[\sin(\theta_o) - \sin(\theta_r)]. \tag{2}$$

Both Eqs. (1) and (2) assume an infinitely thin holographic medium with an interference fringe pattern constrained to the surface. Taking into account thickness of the recording medium is important for faithful characterization of diffraction effects and intensity filtering. Moreover, changes in thickness with shrinkage or swelling of the recording medium can alter the medium's optical properties. Latta [2] derived a general computer-based analysis of thick holograms that decomposes ray vectors into their directional cosines x, y, and z, with y oriented vertically and z oriented perpendicular to the recording medium:

$$x_o' - x_r' = m\frac{\lambda'}{\lambda}\frac{(x_o - x_r)}{M_x}, \tag{3a}$$

$$y_o' - y_r' = m\frac{\lambda'}{\lambda}\frac{(y_o - y_r)}{M_y}, \tag{3b}$$

$$z_o' - z_r' = m\frac{\lambda'}{\lambda}\frac{(z_o - z_r)}{M_z}. \tag{3c}$$

M_x, M_y, and M_z are the magnification scaling factors of the recording medium in the x, y, and z directions. If all vectors are constrained to the yz plane and magnification is considered only in the z direction, then Eq. (3b) reduces to the grating Eq. (2). Eq. (3c) also reduces to the complementary trigonometric equation that takes into account thickness change of the recording material:

$$\cos(\theta_o') - \cos(\theta_r') = m\frac{\lambda'}{\lambda}\frac{\cos(\theta_o) - \cos(\theta_r)}{M_z}. \tag{4}$$

When all three Eqs. (3) [2] or both Eqs. (2) and (4) [3] are simultaneously satisfied, we can determine what wavelengths will be diffracted at what angles. We will assume on-Bragg reconstruction throughout this paper. More sophisticated accounts that include off-Bragg reconstruction are incorporated in Kogelnik's coupled wave theory for thick hologram gratings [4,5], and in Goodman's blurred grating vector analysis that defines the spread of the angular and bandwidth reconstruction [6].

If we were to combine Latta's Eqs. (3) into a single equation resembling Welford's Eq. (1), we would obtain

$$\hat{o}' - \hat{r}' = m\frac{\lambda'}{\lambda}(\hat{o} - \hat{r})\;./\vec{M}, \quad (5)$$

where the ./ operator refers to element-by-element division ($[a,b,c]./\vec{M}=[a/M_x,b/M_y,c/M_z]$). This equation differs from Welford's Eq. (1) by the exclusion of the local surface normal vector $\hat{n}$ and inclusion of the magnification vector $\vec{M}$.

Welford's equation and Latta's equations have been incorporated in various optics design software packages that are widely used today. The author took an interest in their derivations and took the opportunity to relate the two equations by re-deriving one from the other. This paper roughly follows Welford's original approach, with the inclusion of scaling factors to account for changes in the geometry of the recording medium between construction and reconstruction of a hologram.

2. DERIVATION

In the following analysis, we will assume that the length of the unit vectors is many wavelengths and we will consider only Bragg diffraction. Generally held assumptions about holograms include:

1. The recording material operates in a linear region to produce sinusoidal gratings.
2. The hologram is a linear superposition of component planar gratings with independent diffraction effects.

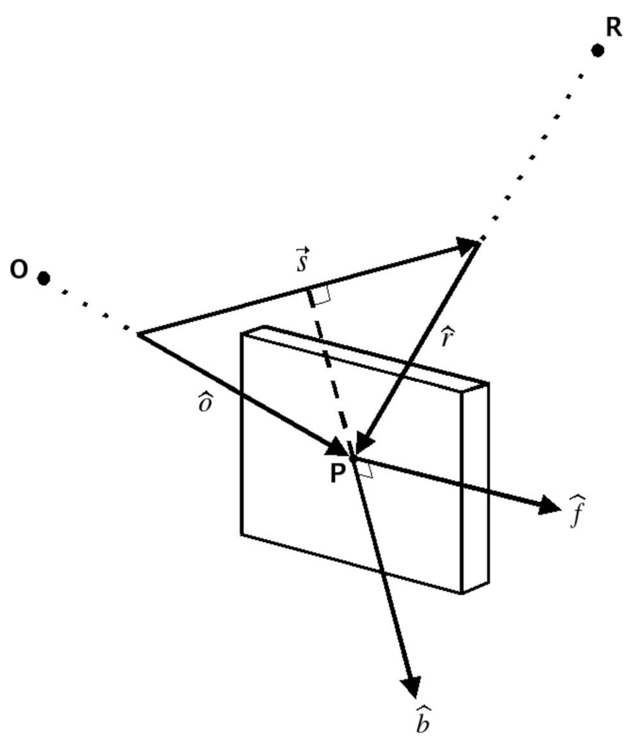

Fig. 1. Construction geometry: rays $\hat{o}$ and $\hat{r}$ emanating from source points O and R intersect at point P in the volume of a holographic recording medium. An interference fringe plane is formed along the bisector $\hat{b}$ to the two rays. The plane also contains a fringe vector $\hat{f}$ such that $\hat{f}$, $\hat{b}$, and $(\hat{o}-\hat{r})$ are perpendicular to each other.

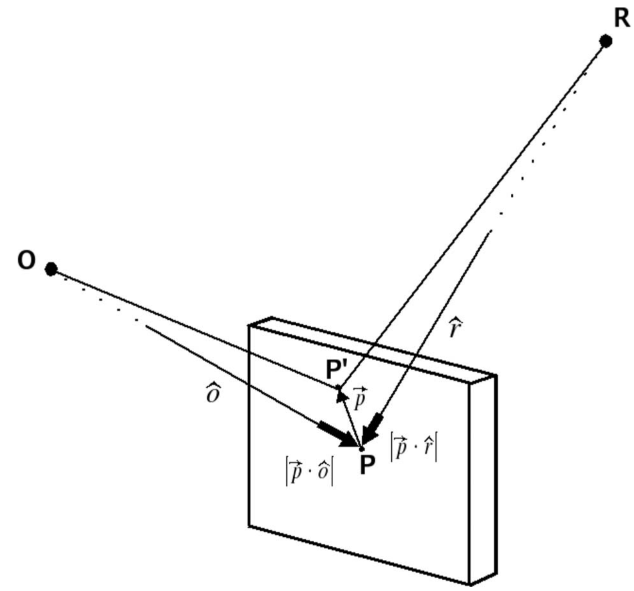

Fig. 2. Constructive interference: a point P′ offset slightly from point P may be the site of constructive interference if the optical path difference is equal to the construction wavelength [see Eq. (7)].

We also assume that changes in index of refraction across the surfaces of the holographic medium would be accounted for by an application of Snell's law [7].

Welford's original approach consisted of two steps for the derivation of his ray-tracing equation. First, he found the direction and spacing of the fringes constructed at an arbitrary point within the recording medium, given a pair of object and reference rays and their wavelength. Second, he calculated the diffracted ray given a reconstruction ray of a given wavelength. We will include an intermediate step to account for the direction and spacing of these fringes after geometrical changes of the recording medium.

Figure 1 illustrates the construction geometry. The object and reference unit vectors are $\hat{o}$ and $\hat{r}$, intersecting at a point P. An interference fringe plane bisects $\hat{o}$ and $\hat{r}$. We will define bisector $\hat{b}$ and fringe $\hat{f}$ vectors to lie in the fringe plane perpendicular to each other:

$$\hat{f} = \hat{b} \times \hat{s}, \quad (6)$$

where $\vec{b}=\hat{o}+\hat{r}$ and $\vec{s}=\hat{o}-\hat{r}$.

If we consider a point P′ displaced a small distance from point P (small relative to OR), we can provide an expression for the optical path difference D illustrated in Fig. 2:

$$D = (\mathrm{OP}' - \mathrm{RP}') - (\mathrm{OP} - \mathrm{RP}) = (\mathrm{OP}' - \mathrm{OP}) + (\mathrm{RP} - \mathrm{RP}')$$
$$\approx \vec{p}\cdot\hat{o} - \vec{p}\cdot\hat{r} \approx \vec{p}\cdot\vec{s}. \quad (7)$$

For an adjacent fringe to be formed at this new location P′, there must be constructive interference, so the smallest optical path difference must be equal to the construction wavelength λ:

$$\vec{p} \cdot \vec{s} \cong \lambda. \tag{8}$$

The interfringe spacing is parallel to $\vec{s}$ and is related to this minimum optical path difference as follows:

$$\vec{p} \cdot \vec{s} = |\vec{s}| \vec{p} \cdot \hat{s} = |\vec{s}| \sigma,$$

$$\sigma = \frac{\lambda}{|\vec{s}|}. \tag{9}$$

The first step of the derivation is complete. We have defined the direction and the spacing of the hologram fringes in Eqs. (6) and (9).

For the second step we will do the same for the fringes after magnification scaling of the recording medium (Fig. 3, with dashed line representing effective thickness [7]).

After magnification (Fig. 3), the reconstruction fringe vector is computed as

$$\vec{f}' = \hat{f} . * \vec{M}, \tag{10}$$

where the $.*$ operator refers to element-by-element multiplication ($[a,b,c].*\vec{M}=[aM_x, bM_y, cM_z]$). The magnified fringe vector may also be expressed in a similar manner as in Eq. (6):

$$\hat{f}' = \hat{b}' \times \hat{s}', \tag{11}$$

where $\vec{b}'=\hat{o}'+\hat{r}'$ and $\vec{s}'=\hat{o}'-\hat{r}'$. In Eq. (6) the construction geometry determines the direction of $\hat{f}$. In Eq. (10) magnification determines the direction of $\hat{f}'$. In Eq. (11) the direction of $\hat{f}'$ constrains the reconstruction geometry.

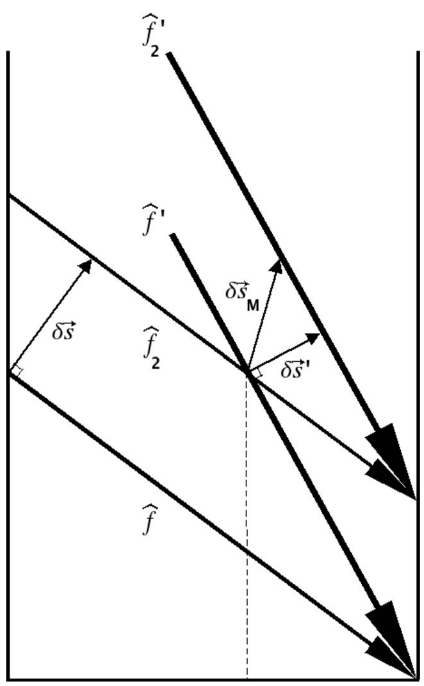

Fig. 3. Magnification: if the holographic recording medium shrinks or swells in one or more directions, the fringe vectors are magnified as well. Pictured here is a pair of fringe vectors before and after shrinkage of the recording medium. As the fringe vectors tilt ($\hat{f}, \hat{f}_2$ to $\hat{f}', \hat{f}_2'$), the interfringe distance along ∂s shrinks to the interfringe distance along $\partial s'$. Although the postshrinkage thickness of the recording medium is not shown, the dashed line marks the effective thickness [7] between points on a fringe vector and corresponding points on the magnified fringe vector in the direction of the magnification scaling vector.

The magnified interfringe spacing σ' may be expressed in a similar form as in Eq. (9). For on-Bragg reconstruction, the fringe plane that would be formed by the rays $\hat{o}'$ and $\hat{r}'$ must either coincide with those actually in the hologram or be harmonics of them. The integer factor m is therefore included:

$$\sigma' = m \frac{\lambda'}{|\vec{s}'|}. \tag{12}$$

Since the interfringe spacing vector $\partial \vec{s}$ of Fig. 3 is perpendicular to the fringe vector $\hat{f}$, magnification has a reciprocal effect as compared with Eq. (10):

$$\partial \hat{s}' = \frac{\partial \vec{s} ./ \vec{M}}{|\partial \vec{s} ./ \vec{M}|}. \tag{13}$$

The magnified interfringe spacing σ' may also be expressed in terms of σ by projecting the magnified interfringe vector ($\partial \vec{s} .* \vec{M}$) onto the reconstruction interfringe vector $\partial \hat{s}'$ and substituting with Eq. (13):

$$\sigma' = |\partial \vec{s}'| = (\partial \vec{s} . * \vec{M}) \cdot \partial \hat{s}' = \frac{(\partial \vec{s} . * \vec{M}) \cdot (\partial \vec{s} ./ \vec{M})}{|\partial \vec{s} ./ \vec{M}|}$$

$$= \frac{|\partial \vec{s}|^2}{|\partial \vec{s} ./ \vec{M}|} = \frac{|\partial \vec{s}|}{|\partial \hat{s} ./ \vec{M}|} = \frac{\sigma}{|\hat{s} ./ \vec{M}|}. \tag{14}$$

For the final step we will combine the previous results to express on-Bragg reconstruction through the magnified recording medium. From Eqs. (10) and (11) we may write

$$\hat{b}' \times \hat{s}' = \frac{\hat{f} . * \vec{M}}{|\hat{f} . * \vec{M}|}. \tag{15}$$

Multiplying the right side by $|\vec{s}|/|\vec{s}|$ and expanding $\hat{f}$,

$$\hat{b}' \times \hat{s}' = \frac{|\vec{s}|}{|\vec{s}|} \frac{(\hat{b} \times \hat{s}) . * \vec{M}}{|(\hat{b} \times \hat{s}) . * \vec{M}|}, \tag{16}$$

enables us to substitute σ/λ for $1/|\vec{s}|$ from Eq. (9):

$$\hat{b}' \times \hat{s}' = \frac{\sigma}{\lambda} \frac{(\hat{b} \times \vec{s}) . * \vec{M}}{|(\hat{b} \times \hat{s}) . * \vec{M}|}. \tag{17}$$

Likewise, multiplying both sides by $|\vec{s}'|$,

$$\hat{b}' \times \vec{s}' = |\vec{s}'| \frac{\sigma}{\lambda} \frac{(\hat{b} \times \vec{s}) . * \vec{M}}{|(\hat{b} \times \hat{s}) . * \vec{M}|}, \tag{18}$$

enables us to substitute $m\lambda'/\sigma'$ for $|\vec{s}'|$ from Eq. (12):

$$\hat{b}' \times \vec{s}' = m \frac{\lambda' \sigma}{\lambda \sigma'} \frac{(\hat{b} \times \vec{s}) . * \vec{M}}{|(\hat{b} \times \hat{s}) . * \vec{M}|}. \tag{19}$$

Substituting Eq. (14) and (19) results in a generalization of Welford's equation that includes scaling changes of the recording medium:

$$\hat{b}' \times \vec{s}' = m\frac{\lambda'}{\lambda}\frac{(\hat{b} \times \vec{s}) .* \vec{M}}{|(\hat{b} \times \hat{s}) .* \vec{M}|}|\hat{s} ./\vec{M}|. \tag{20}$$

As with Welford's and Latta's equations, applying Snell's law will account for changes in index of refraction across the surfaces of the holographic medium and between the time of construction and reconstruction [8].

3. REDUCTION TO COMMON RAY-TRACING EQUATIONS

We may reduce Eq. (16) to standard ray-tracing equations. If there is no magnification ($\vec{M}=[1,1,1]$), and the bisector is replaced by the local surface normal unit vector to constrain the interference pattern to a surface grating ($\hat{b}$ and $\hat{b}'$ are set equal to $\hat{n}$), then the general ray-tracing equation reduces to Welford's Eq. (1) and in turn to the grating Eq. (2).

We may also derive Eq. (5) (Latta's equations) by applying the steps used in Eqs. (17)–(20) to $\hat{s}'$. Since vector $\hat{s}'$ is parallel to $\partial\hat{s}'$, we may put it in the form of Eq. (13):

$$\hat{s}' = \frac{\vec{s} ./\vec{M}}{|\vec{s} ./\vec{M}|}. \tag{21}$$

Expanding the denominator, so,

$$\hat{s}' = \frac{1}{|\vec{s}|}\frac{\vec{s} ./\vec{M}}{|\hat{s} ./\vec{M}|} \tag{22}$$

enables us to substitute σ/λ for $1/|\vec{s}|$ from Eq. (9):

$$\hat{s}' = \frac{\sigma}{\lambda}\frac{\vec{s} ./\vec{M}}{|\hat{s} ./\vec{M}|}. \tag{23}$$

Likewise, multiplying both sides by $|\vec{s}'|$, so,

$$\vec{s}' = |\vec{s}'|\frac{\sigma}{\lambda}\frac{\vec{s} ./\vec{M}}{|\hat{s} ./\vec{M}|} \tag{24}$$

enables us to substitute $m\lambda'/\sigma'$ for $|\vec{s}'|$ from Eq. (12):

$$\vec{s}' = m\frac{\lambda'\sigma}{\lambda\sigma'}\frac{\vec{s} ./\vec{M}}{|\hat{s} ./\vec{M}|}. \tag{25}$$

Finally, substituting Eq. (14) allows us to cancel terms, and—remembering that $s=\hat{o}-\hat{r}$—yields Latta's Eq. (5):

$$\vec{s}' = m\frac{\lambda'}{\lambda}\vec{s} ./\vec{M}. \tag{26}$$

4. CONCLUSIONS

The general hologram ray-tracing equation derived in this paper allows for arbitrary ray directions and wavelengths during construction and reconstruction, accounts for changes in the shape of the recording medium, and reduces to well-accepted ray-tracing analyses for thick and thin holograms. Specifically, the author has derived Latta's ray-tracing equations for thick holograms from Welford's equation for thin holograms of arbitrary shape.

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